JEE - ADVANCED

2007 onwards ...

Author ⇔ Ankur Roy Choudhury

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"Fully - Solved"

CONTENTS

Series (AP, G.P, HP)

1) If the sum of first 'n' terms of an A.P. is cn^2, then the sum of squares of these n terms is (2009)

(A) $\frac{n(4n^2-1)c^2}{6}$ B) $\frac{n(4n^2+1)c^2}{3}$ ✓ C) $\frac{n(4n^2-1)c^2}{3}$ D) $\frac{n(4n^2+1)c^2}{6}$

Sol^n: $T_n = S_n - S_{n-1} = cn^2 - c(n-1)^2 = c[n^2-(n-1)^2] = c[2n-1]$

$\therefore \sum_{r=1}^{n} T_r^2 = c^2\sum_{r=1}^{n}[4r^2-4r+1] = c^2\left[4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1\right]$

$= c^2\left[\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right] = c^2 n\left[\frac{4(n+1)(2n+1)}{6} - \frac{4(n+1)}{2} + 1\right]$

$= c^2\left[\frac{8n^2+12n+4-12n-12+6}{6}\right] = c^2\left[\frac{8n^2-2}{6}\right]n$

$= c^2 n\left[\frac{4n^2-1}{3}\right]$

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2) Let a_1, a_2, a_3 be in H.P with $a_1 = 5$ & $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is (2012)

1) 22 2) 23 3) 24 ✓4) 25.

Sol^n: a_1, a_2, a_3 are in H.P $\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ are in AP.

$\therefore \frac{1}{a_n} = \frac{1}{a_1} + (n-1)d$; $\because \frac{1}{a_n} < 0 \therefore \frac{1}{a_1} + (n-1)d < 0$

$d = \frac{\frac{1}{25}-\frac{1}{5}}{19} = \frac{-4}{19\times25}$; $\therefore \frac{1}{5} + (n-1)\left(\frac{-4}{19\times25}\right) < 0$

$\Rightarrow \frac{1}{5} - \frac{4(n-1)}{19\times25} < 0 \Rightarrow \frac{1}{5} < \frac{4(n-1)}{19\times25}$ $\Rightarrow n > \frac{19\times5}{4} + 1 \Rightarrow n \geq 24.9$

$\Rightarrow n \geq 25$.

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3) Let $S_n = \sum_{K=1}^{4n} (-1)^{\frac{K(K+1)}{2}} \times K^2$. Then S_n can take values (2013)

① 1056 ② 1088 ③ 1120 ④ 1332

Sol^n: $S_n = (-1)^1 1^2 + (-1)^3 2^2 + (-1)^6 3^2 + (-1)^{10} 4^2 + \cdots$ upto 4n terms

$= -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + \cdots + (4n)^2$

$= -1 + 5 - 9 + 13 - 17 + \cdots + 16n^2$

$= \underbrace{4+4+4+4+\cdots}_{n\text{ times}} + 16n^2 = 4n + 16n^2 = 4n(4n+1)$

4> Let V_r denote the sum of first r terms of an A.P whose first term is r & the common difference is $2r-1$. Let $T_r = V_{r+1} - V_r - 2$, $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots\ldots$ (2007)

a) The sum $V_1 + V_2 + \ldots\ldots + V_m =$

i) $\frac{1}{12} m(m+1)(3m^2 - m + 1)$ iii) $\frac{1}{2} m(2m^2 - m + 1)$

✓ii) $\frac{1}{12} m(m+1)(3m^2 + m + 2)$ iv) $\frac{1}{3}(2m^3 - 2m + 3)$

b) T_r is always

i) An odd no. iii) A prime no.

ii) An even no. ✓iv) A composite no.

c) Which one of the following is a correct statement?

i) $Q_1, Q_2, Q_3 \ldots$ are in A.P with common difference 5

ii) $Q_1, Q_2, Q_3 \ldots$ are in A.P with common difference 6

iii) $Q_1, Q_2, Q_3 \ldots$ are in A.P with common difference 11.

iv) $Q_1 = Q_2 = Q_3$.

Sol^n: a) $V_r = \frac{r}{2}\left[2r + (r-1)(2r-1)\right] = r^2 + \frac{r}{2}(2r^2 - 3r + 1)$

$\Rightarrow V_r = r^2 + r^3 - \frac{3}{2}r^2 + \frac{r}{2} \Rightarrow V_r = r^3 - \frac{r^2}{2} + \frac{r}{2}$

$\therefore \sum_{r=1}^{m} V_r = \sum_{r=1}^{m} r^3 - \frac{1}{2}\sum_{r=1}^{m} r^2 + \frac{1}{2}\sum_{r=1}^{m} r = \left[\frac{m(m+1)}{2}\right]^2 - \frac{1}{2}\cdot\frac{m(m+1)\cdot(2m+1)}{6} + \frac{m(m+1)}{4}$

$= \frac{m(m+1)}{4}\left[m(m+1) - \frac{2m+1}{3} + 1\right] = \frac{m(m+1)}{12}\left[3m^2 + 3m - 2m - 1 + 3\right]$

$= \frac{m(m+1)}{12}\left[3m^2 + m + 2\right]$ Ans.

b) $T_r = V_{r+1} - V_r - 2 = \left[(r+1)^3 - \frac{(r+1)^2}{2} + \frac{(r+1)}{2}\right] - \left[r^3 - \frac{r^2}{2} + \frac{r}{2}\right] - 2$

$= \left(\frac{r+1}{2}\right)\left[2(r+1)^2 - (r+1) + 1\right] - \left[r^2 - \frac{r}{2} + \frac{1}{2}\right]r - 2$

$= \left(\frac{r+1}{2}\right)\left[2r^2 + 2 + 4r - r - 1 + 1\right] - \left[r^3 - \frac{r^2}{2} + \frac{r}{2}\right] - 2$

$\Rightarrow \frac{r+1}{2}\left[2r^2 + 3r + 2\right] - \left[\frac{2r^3 - r^2 + r}{2}\right] - 2 = \frac{2r^3 + 3r^2 + 2r + 2r^2 + 3r + 2 - 2r^3 + r^2 - r - 4}{2}$

$\Rightarrow \frac{6r^2 + 4r - 2}{4} = \frac{\cancel{2}(3r^2 + 2r - 1)}{\cancel{2}} = (r+1)(3r-1)$

Suppose: $r = 2$, $\therefore T_r = 3 \times 5 = 15$
$r = 3$; $\therefore T_r = 4 \times 8 = 32$ $\Bigg] \Rightarrow$ Both are composite nos. $\therefore T_r$ is composite no.

c) $Q_r = T_{r+1} - T_r = \left[3(r+1)^2 + 2(r+1) - 1\right] - \left[3r^2 + 2r - 1\right] = 6r + 5$

$Q_1 = 11$, $Q_2 = 17$, $Q_3 = 23$ $\Rightarrow$ common diff 6 $\Rightarrow$ A.P.

5> Let A_1, G_1, H_1 denote A.M, G.M, H.M respectively of 2 distinct positive nos. For $m \geq 2$, let A_{m-1} & H_{m-1} have AM, GM, HM as A_m, G_m, H_m respectively (2007)

a) Which one of the following statements is correct? (2007)

i) $G_1 > G_2 > G_3 > \cdots$ ✓iii) $G_1 = G_2 = G_3$

(ii) $G_1 < G_2 < G_3 < \cdots$ iv) $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_4 > G_6 > \cdots$

b) Which one of the following statements is correct?

✓i) $A_1 > A_2 > A_3 > \cdots$ iii) $A_1 > A_3 > A_5 \ldots$ and $A_2 < A_4 < A_6 < \cdots$

ii) $A_1 < A_2 < A_3 < \cdots$ iv) $A_1 < A_3 < A_5 \cdots$ and $A_2 > A_4 > A_6 > \cdots$

c) Which one of the following statements is correct?

i) $H_1 > H_2 > H_3 > \cdots$ (ii) $H_1 < H_2 < H_3 < \cdots$ ✓

iii) $H_1 > H_3 > H_5 > \cdots$ and $H_2 < H_4 < H_6 < \cdots$

iv) $H_1 < H_3 < H_5 < \cdots$ and $H_2 > H_4 > H_6 > \cdots$

Sol^n: a) According to the problem, Suppose there are 8 nos.

Let these nos. be $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$.

i.e $n = 8$. Let these nos. be

$\therefore$ There will be 6 AMs, 6 GMs, 6 HMs.

Let these means be represented by:

$A_1 \ A_2 \ A_3 \ A_4 \ A_5 \ A_6 \Rightarrow$ AMs

$G_1 \ G_2 \ G_3 \ G_4 \ G_5 \ G_6 \Rightarrow$ GMs

$H_1 \ H_2 \ H_3 \ H_4 \ H_5 \ H_6 \Rightarrow$ H.Ms.

Again b/w A_5 and H_5 there will be one AM, one GM, one H.M & it is applicable for all other cases i.e (A_1, H_1), (A_2, H_2), ……

$\therefore$ In this case, A_6, G_6, H_6 are 3 means.

$\therefore G_{K+1} = \sqrt{A_K H_K}$; $H_{K+1} = \frac{2A_K H_K}{A_K + H_K}$; $A_{K+1} = \frac{A_K + H_K}{2}$

(as $G_6 = \sqrt{A_5 H_5}$) (as $H_6 = \frac{2A_5H_5}{A_5+H_5}$) (As $A_6 = \frac{A_5+H_5}{2}$)

(i) $\therefore G_K^2 = A_K H_K$ (As G.M b/w two nos, a, b is $\sqrt{ab}$).

$\therefore G_K^2 = G_{K+1}^2 \Rightarrow G_K = G_{K+1} \Rightarrow G_1 = G_2 = G_3 \cdots$

(ii) $A_{K+1} = \frac{A_K + H_K}{2} < \frac{A_K + A_K}{2} \Rightarrow A_{K+1} < A_K \Rightarrow A_2 < A_1$

(iii) $H_{K+1} = \frac{2A_K H_K}{A_K + H_K} = \frac{A_K H_K}{\frac{A_K + H_K}{2}} = \frac{A_K \cdot H_K}{A_{K+1}} \Rightarrow \frac{H_{K+1}}{H_K} = \frac{A_K}{A_{K+1}} > 1$

$\therefore H_{K+1} > H_K$

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5) Suppose 4 distinct +ve nos. a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$, and $b_4 = b_3 + a_4$ (2008)

STATEMENT-1 Nos. b_1, b_2, b_3, b_4 are neither in A.P nor in G.P

STATEMENT-2 Nos. b_1, b_2, b_3, b_4 are in H.P

(i) Statement 1 is True. Statement 2 is true. Stmt. 2 is correct explanation for Stmt-1.

(ii) Statement 1 is True. Statement 2 is NOT a correct explanation for Stmt-1.

✓(iii) Statement 1 is true, statement-2 is false. (iv) Statement-1 is false & statement 2 is true.

7) Let S_K, $K = 1, 2, 3 \ldots 100$, denote the sum of infinite geometric series whose first term is $\frac{K-1}{K!}$ and common ratio is $1/K$. Then the value of $\frac{100^2}{100!} + \sum_{K=1}^{100} |(K^2-3K+1)S_K|$ is (2010)

Sol^n: $a = \frac{K-1}{K!}$; $S_K = \frac{a}{1-r} = \frac{\frac{K-1}{K!}}{1-1/K} = \frac{1}{(K-1)!}$

$\therefore \frac{100^2}{100!} + \sum_{K=1}^{100} \left|(K^2-3K+1)\frac{1}{(K-1)!}\right| = \frac{100^2}{100!} + 1 + \sum_{K=2}^{100} \left|\frac{(K-1)^2-K}{(K-1)!}\right|$

$= \frac{100^2}{100!} + 1 + \sum_{K=2}^{100}\left(\frac{K-1}{(K-2)!} - \frac{K}{(K-1)!}\right) = \frac{100^2}{100!} + 1 + 1 \sum_{K=3}^{100}\left[\frac{K-1}{(K-2)!} - \frac{K}{(K-1)!}\right]$

$= \frac{100^2}{100!} + 1 + 1\left(\frac{2}{1!} - \frac{3}{2!}\right) + \left(\frac{3}{2!} - \frac{4}{3!}\right) + \left(\frac{4}{3!} - \frac{5}{4!}\right) + \cdots + \frac{99}{98!} - \frac{100}{99!}$

$= 100 + 1 + 1 + 1 - \frac{100^2}{100!} = 3$ Ans.

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8) Let $a_1, a_2, a_3, \ldots a_{11}$ be real nos. satisfying $a_1 = 15$, $27 - 2a_2 > 0$. and $a_K = 2a_{K-1} - a_{K-2}$ for $K = 3, 4, \ldots 11$. If $\frac{a_1^2+a_2^2+\cdots+a_{11}^2}{11} = 90$ (2010) then the value of $\frac{a_1+a_2+a_3+\cdots+a_{11}}{11} =$

Sol: $\because a_K + a_{K-2} = 2a_{K-1}$, $\Rightarrow a_1, a_2, a_3 \ldots a_{11}$ are in A.P.

Now; $a_1^2 + a_2^2 + a_3^2 + \cdots + a_{11}^2 = 15^2 + (15+d)^2 + (15+2d)^2 + \cdots (15+10d)^2$

$= (15^2) + (15^2 + 2\cdot15\cdot d + d^2) + (15^2 + 2\cdot15\cdot2\cdot d + 2^2d^2) + \cdots + (15^2 + 2\cdot15\cdot10d + 10^2\cdot d^2)$

$= \underbrace{(15^2+15^2\cdots+15^2)}_{11 \text{ terms}} + 2\cdot15\cdot d(1+2+3+\cdots+10) + (1^2+2^2+3^2+\cdots+10^2)d^2$

$= 15^2 \times 11 + 2\cdot15\cdot d \times \frac{10(10+1)}{2} + \frac{10\cdot(10+1)(2\cdot10+1)}{6}$

$= 11(15^2 + 150d + 35d^2)$. $\because \frac{a_1^2+a_2^2+\cdots+a_{11}^2}{11} = 90$

$\therefore \frac{(15^2+150d+35d^2)11}{11} = 90 \Rightarrow 7d^2 + 30d + 45 = 18 \Rightarrow d = -3, -9/7$.

given $a_2 < \frac{27}{2} \Rightarrow a_2 < 13.5$; so if $d = -3$; $a_2 = 15 - 3 = 12 < 13$
$d = -9/7$, $a_2 = 15 - \frac{9}{7} > 13.5$

$\therefore d = -3$. $\therefore \frac{a_1+a_2+a_3+\cdots+a_{11}}{11} = \frac{15+(15+d)+(15+2d)\cdots+(15+10d)}{11}$

$= \frac{15\times11 + \frac{10\times11}{2}\times d}{11} = 15 + (5\times-3) = 0$ Ans.

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9) Let $a_1, a_2, a_3, \ldots a_{100}$ be an A.P with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i$, $1 \le p \le 100$. For any integer n with $1 \le n \le 20$. Let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then $a_2 =$ (2011)

Sol^n: $a_1 = 3$; $S_p = a_1 + a_2 + \cdots + a_p = \frac{p}{2}[2a_1 + (p-1)d]$ $\because \frac{S_m}{S_n}$ does not depend on n

$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}[6+(5n-1)d]}{\frac{n}{2}[6+(n-1)d]} = \frac{5[6-d+5nd]}{[6-d+nd]}$

$\therefore 6 - d = 0 \Rightarrow d = 6$. $\therefore a_2 = 3 + 6 = 9$ Ans

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10) The minimum value of sum of real nos. $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is ___

Sol^n: $\because a > 0$, let $a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1 = S$

clearly $s = 8$ if $a > 1$.
For all values of $a \in R^{+} - \{1\}$, $s > 8$
$\therefore$ minimum value of $s = 8$.

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11) A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the nos. on the remaining cards is 1224. If smaller of the nos. on the removed cards is K, then $K - 20 =$ (2013)

Soln: clearly $(1+2+3+\dots+n-2) \le 1224 \le (3+4+\dots+n)$

$$\Rightarrow \frac{(n-2)(n-1)}{2} \le 1224 \le \frac{n-2}{2}(3+n)$$

$\Rightarrow n^2 - 3n - 2446 \le 0$ & $n^2 + n - 2454 \ge 0$

$n = \frac{+3 \pm \sqrt{9+9784}}{2}$

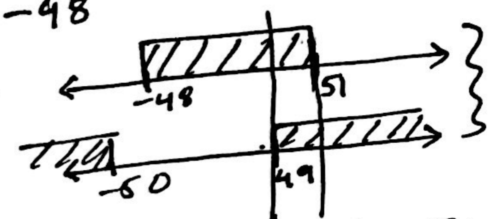
$n = +\frac{3 \pm 98.95}{2} = +\frac{3 \pm 99}{2}$

$n \approx -48, 51 \Rightarrow (n-51)(n+48) \le 0$

$n = \frac{-1 \pm \sqrt{1+9816}}{2} = \frac{-1 \pm 99.08}{2} = \frac{-1 \pm 99}{2}$

$n \approx 49, -50 \Rightarrow (n-49)(n+50) \ge 0$

$\therefore$



$\Rightarrow 49 < n < 51 \Rightarrow n = 50$.
$\because$ Smaller of the nos. on removed card is K, other will be $K+1$

$\therefore \frac{50(50+1)}{2} - (K+K+1) = 1224 \Rightarrow K = 25 \Rightarrow K - 20 = 5$ Ans.

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12) Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in G.P and A.M of a, b, c is $b+2$, then value of $\frac{a^2+a-14}{a+1}$ is . (2014)

Soln: $\because \frac{b}{a}$ is integer & a, b, c are in GP

$\therefore \frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$; Now $\frac{a+b+c}{3} = b+2 \Rightarrow a+b+c = 3b+6$

$\Rightarrow a - 2b + c = 6 \Rightarrow a - 2b + \frac{b^2}{a} = 6 \Rightarrow a^2 - 2ab + b^2 = 6a$

$\Rightarrow (a-b)^2 = 6a \Rightarrow (1 - b/a)^2 = \frac{6}{a} \Rightarrow \left(\frac{b}{a} - 1\right)^2 = \frac{6}{a}$; $\because$ LHS is an posit. integer, $\therefore \frac{6}{a}$ has to be +ve integer. $\Rightarrow a$ can take val.

$\because \frac{6^2+6-14}{6+1} = \frac{36+6-14}{7} = \frac{42-14}{7} = \frac{28}{7} = 4$ Ans.

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13) Suppose that all the terms of an A.P are natural nos. If the ratio of the sum of the first seven terms to the sum of first eleven terms is 6 : 11 and the seventh term lies in b/w 130 & 140, then common diff of this AP is ___ (2015)

Paper - 2 - 2018

One or More type

1) For any positive integer n, define $f_n : (0,\infty) \to R$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \ \forall\, x \in (0,\infty).$$ Here

$\tan^{-1}x \in (-\pi/2, \pi/2)$. Then which of the following statements are true?

A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$ (B) $\sum_{j=1}^{10} (1+f'(0))\sec^2(f_j(0)) = 10$

c) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

Sol: $f_n(x) = \tan^{-1}\left(\frac{1}{1+(x+1)\cdot x}\right) + \tan^{-1}\left(\frac{1}{1+(x+2)(x+1)}\right) + \dots + \tan^{-1}\left(\frac{1}{1+(x+n)(x+n-1)}\right)$

$f_n(x) = \tan^{-1}\left(\frac{x+1-x}{1+(x+1)x}\right) + \tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right) + \dots + \tan^{-1}\left(\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right)$

$\Rightarrow f_n(x) = \tan^{-1}\cancel{(x+1)} - \tan^{-1}x + \tan^{-1}\cancel{(x+2)} - \tan^{-1}\cancel{(x+1)} + \dots + \tan^{-1}(x+n) - \tan^{-1}\cancel{(x+n-1)}$

$\Rightarrow f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$

$f'_n(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$, $f_n(0) = \tan^{-1}n \Rightarrow \tan^2(\tan^{-1}n) = n^2$

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = \sum_{j=1}^{5} j^2 = \frac{5\times 6\times 11}{6} = 55$

(B) $f'_n(0) = \frac{1}{1+n^2} - 1 \Rightarrow 1 + f'_n(0) = \frac{1}{1+n^2}$

$\sec^2(f_n(0)) = \sec^2(\tan^{-1}n) = \sec^2\sec^{-1}\sqrt{1+n^2} = 1+n^2$

$\therefore (1+f'_n(0))\sec^2 f_n(0) = \frac{1}{1+n^2} \times 1+n^2 = 1$.

$\therefore \sum_{j=1}^{10} (1+f'(0))(\sec^2(f_j(0))) = \sum 1 = 10$.

(C) $\lim_{x\to\infty} \tan f_n(x) = \lim_{x\to\infty} \tan\left[\tan^{-1}(x+n) - \tan^{-1}x\right] = \lim_{x\to\infty} \tan\tan^{-1}\frac{n}{1+(x+n)x}$

$= \lim_{x\to\infty} \frac{n}{1+(x+n)x} = \frac{n}{\infty} = 0$

(D) $\lim_{x\to\infty} \sec^2\tan^{-1}\frac{n}{1+x^2+nx} = \lim_{x\to\infty} \sec^2\sec^{-1}\left(\frac{\sqrt{n^2+x^2+nx+1}}{nx+1+x^2}\right)$

$= \lim_{x\to\infty} \frac{n^2+x^2+nx+1}{x^2+nx+1} \Rightarrow \lim_{x\to\infty} \frac{\cancel{x^2}\left(\frac{n^2}{x^2}+1+\frac{n}{x}+1/x^2\right)}{\cancel{x^2}\left(1+n/x+1/x^2\right)} = \frac{1}{1} =$

$\therefore$ A, B, C, D Ans.

27) Let T be the line passing through the points $P(-2,7)$ and $Q(2,-5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 'touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all st. line segments joining a pair of distinct points of E_1 and passing through the point $R(1,1)$ be F_2. Let E_2 be the set of the midpoints of the line segments in the set F_2. Then which of the following are true?

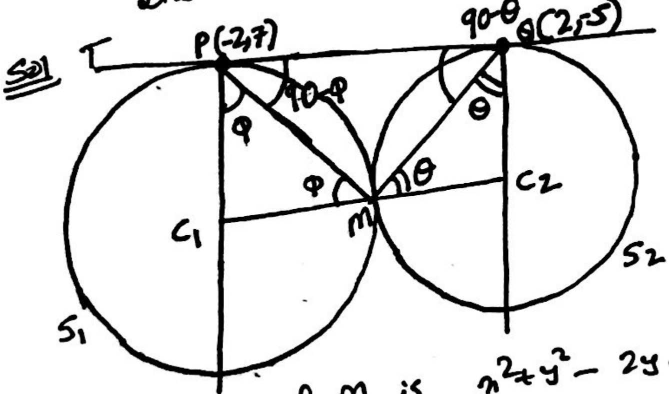
(A) $(-2,7)$, lies on E_1

(B) Point $\left(\frac{-4}{5}, \frac{7}{5}\right)$ does not lie on E_2

(C) Point $(1/2, 1)$ lies at E_2

(D) Point $(0, 3/2)$ does not lie on E_1.

Sol



$\angle QC_2M + \angle PC_1M = \pi$

$\Rightarrow \pi - 2\theta + \pi - 2\varphi = \pi$

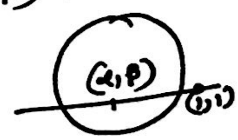
$\Rightarrow \theta + \varphi = \pi/2$

Hence locus of variable point M is $(x+2)(x-2) + (y-7)(y+5) =$ which implies it is a circle with PQ as dia.

∴ locus of M is E_1.

∴ locus of M is $x^2 + y^2 - 2y - 39 = 0$ --- (1)

Equation of chord of circle whose midpoint is (α, β) is $S_1 = T$



$\Rightarrow x\alpha + y\beta - (y+\beta) - 39 = \alpha^2 + \beta^2 - 2\beta - 39$

It passes through $(1,1)$ ⇒ $\alpha^2 + \beta^2 - 2\beta - \alpha + 1 = 0$

∴ locus of midpoint: $x^2 + y^2 - x - 2y + 1 = 0$ ---- (2)

Option A is in-correct although it satisfies equation (1) otherwise line T would touch the S_2 at two points. Also $(4/5, 7/5)$ satisfies equation (2) but again in this case one end of the chord would be $(-2,7)$ which is not included in E_1. ∴ $(4/5, 7/5)$ does not lie in E_2. $(1/2, 1)$ does not satisfy equation (2) ∴ it does not lie in E_2. $(0, 3/2)$ does not satisfy (1), so does not lie on E_1.

So B, D is ans.

(3) Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in R$ and the system of equation (in real variables) $-x+2y+5z=b_1$, $2x-4y+3z=b_2$, $x-2y+2z=b_3$ has at least one solution. Then which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(A) $x+2y+3z=b_1$, $4y+5z=b_2$, and $x+2y+6z=b_3$

(B) $x+y+3z=b_1$, $5x+2y+6z=b_2$ and $-2x-y-3z=b_3$

(C) $-x+2y-5z=b_1$, $2x-4y+10z=b_2$ and $x-2y+5z=b_3$?

(D) $x+2y+5z=b_1$, $2x+3z=b_2$ and $x+4y-5z=b_3$

Sol: $\Delta = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 0$ So for at least one solution, $\Delta_1 = \Delta_2 = \Delta_3 = 0$

Thus $\Delta_1 = \begin{vmatrix} b_1 & 2 & 5 \\ b_2 & -4 & 3 \\ b_3 & -2 & 2 \end{vmatrix} = 0$ $b_1 + 7b_2 = 13b_3$ ....(1)

option A; $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 2 & 6 \end{vmatrix} \neq 0 \Rightarrow$ Unique solution.

D; $\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix} \neq 0 \Rightarrow$ unique solution.

option C: $\Delta = \begin{vmatrix} -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{vmatrix} = 0$ Now $\left.\begin{array}{l} x-2y+5z=-b_1 \\ x-2y+5z=\frac{b_2}{2} \\ x-2y+5z=b_3 \end{array}\right] \Rightarrow$ Parallel planes

So they must be coincident $\Rightarrow -b_1 = \frac{b_2}{2} = b_3$

Now all b_1, b_2, b_3 obtained from equation (1) may not satisfy this relation. So option C is wrong.

option B) $\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{vmatrix} = 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Delta_2 = \begin{vmatrix} 1 & b_1 & 1 \\ 5 & b_2 & 2 \\ 2 & b_3 & 1 \end{vmatrix} = 0 \Rightarrow b_2 - 2b_3 - (5-4)b_1 + 1(5b_3 - 2b_2) = 0$

$\Rightarrow -b_2 - b_1 + 3b_3 = 0$ which does not satisfy eqn (1) for all $b_1, b_2, b_3 \in S$. option b is wrong.

$\therefore$ Ans. A, D.

4) Consider two st. lines, each of which is tangent to both the circle $x^2+y^2 = 1/2$ and the parabola $y^2 = 4x$. Let these lines intersect at point Q. Consider the ellipse whose centre is at the origin O(0,0) and whose semi major axis is OQ.